# A Level Mathematics B (MEI)

**H640/01** MEI Pure Mathematics and Mechanics

Pure

## Question Set 4

1. Simplify $\left(\frac{27}{x^9}\right)^{\frac{2}{3}} \times \left(\frac{x^4}{9}\right)$. [2]

$$\frac{(27)^{2/3}}{(x^9)^{2/3}} = \frac{(\sqrt[3]{27})^2}{x^{9\times 2/3}} = \frac{\cancel{9}}{x^6} \times \frac{x^4}{\cancel{9}} = \frac{x^4}{x^6} = x^{4-6} = x^{-2}$$

$$= \frac{1}{x^2}$$

2. Express $\frac{a+\sqrt{2}}{3-\sqrt{2}}$ in the form $p+q\sqrt{2}$, giving $p$ and $q$ in terms of $a$. [3]

$$\frac{a+\sqrt{2}}{3-\sqrt{2}} \times \frac{(3+\sqrt{2})}{(3+\sqrt{2})} = \frac{3a+a\sqrt{2}+3\sqrt{2}+2}{7}$$

$$= \frac{3a+2}{7} + \frac{(a+3)}{7}\sqrt{2}$$

3. The points A and B have position vectors $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 4 \\ 8 \end{pmatrix}$ respectively.

Show that the exact value of the distance AB is $\sqrt{101}$. [3]

$$\vec{BA} = \begin{bmatrix} -1 \\ 4 \\ 8 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 9 \end{bmatrix}$$

$$|\vec{BA}| = \sqrt{4^2+2^2+9^2}$$

$$|\vec{BA}| = \sqrt{101}$$

4. Find the second derivative of $(x^2+5)^4$, giving your answer in factorised form. [5]

$U = x^2+5$

$V = U^4$

$\frac{dV}{dU} = 4U^3$

$4 \times 2x(x^2+5)^3$

$8x(x^2+5)^3$

$U = 8x$

$V = (x^2+5)^3$

$\frac{dU}{dx} = 8$

$\frac{dV}{dx} = 6x(x^2+5)^2$

$8(x^2+5)^3 + 48x^2(x^2+5)^2$

$= 8(x^2+5)^2(7x^2+5)$

5. **In this question you must show detailed reasoning.**

The function f(x) is defined by $f(x) = x^3 + x^2 - 8x - 12$ for all values of $x$.

a) Use the factor theorem to show that $(x+2)$ is a factor of $f(x)$. [2]

$f(-2) = 0$

$(-2)^3 + (-2)^2 - (8x-2) - 12 = 0$

b) Solve the equation $f(x) = 0$. **[4]**

$$\begin{array}{r|l} & x^2 - x - 6 \\ x+2 & x^3 + x^2 - 8x - 12 \\ & x^3 + 2x^2 \\ \hline & -x^2 - 8x \\ & -x^2 - 2x \\ \hline & -6x - 12 \\ & -6x - 12 \\ \hline & 0 \end{array}$$

$x^2 - x - 6$

$(x+2)(x-3)(x+2)$

$x = -2$

$x = 3$

6. Fig. 6.1 shows the cross-section of a straight driveway 4 m wide made from tarmac.

**Fig. 6.1**

The height $h$ m of the cross-section at a displacement $x$ m from the middle is modelled by $h = \frac{0.2}{1+x^2}$ for $-2 \leqslant x \leqslant 2$.

A lower bound of $0.3615\,\text{m}^2$ is found for the area of the cross-section using rectangles as shown in Fig. 6.2.

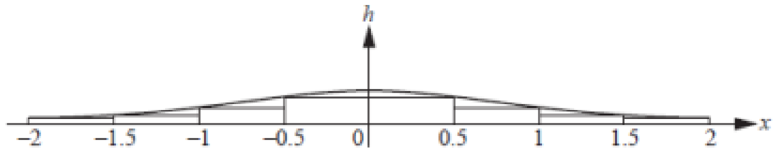


**Fig. 6.2**

a) Use a similar method to find an upper bound for the area of the cross-section. **[3]**

| $x$ | $h$ |
|---|---|
| $0$ | $0.2$ |
| $\pm 0.5$ | $0.16$ |
| $\pm 1$ | $0.1$ |
| $\pm 1.5$ | $4/65$ |
| $\pm 2$ | $0.04$ |

$= 2 \times 0.5(0.2 + 0.16 + 0.1 + \frac{4}{65} + 0.04)$

$= 0.522\ (3sf)$

**(b)** Use the trapezium rule with 4 strips to estimate $\int_0^2 \frac{0.2}{1+x^2}\,dx$. **[2]**

$\text{Area} = \frac{0.5}{2}(0.2 + 0.04 + 2(0.16 + 0.1 + \frac{4}{65}))$

$= 0.221\ (3sf)$

**(c)** The driveway is 10 m long. Use your answer in part **(b)** to find an estimate of the volume of tarmac needed to make the driveway. [2]

$$\text{Volume} = 2 \times (0.221 \times 10)$$
$$= 4.42\text{m}^3$$

**7** **In this question you must show detailed reasoning.**

Fig. 7 shows the curve given parametrically by the equations $x = \frac{1}{t^2}$, $y = \frac{1}{t^3} - \frac{1}{t}$, for $t > 0$.

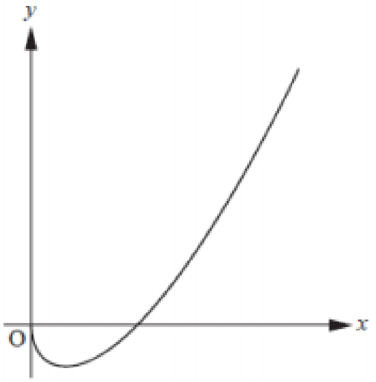


**Fig. 7**

**(a)** Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3-t^2}{2t}$. [3]

$$\frac{dy}{dt}\left(t^{-3} - t^{-1}\right) = -3t^{-4} + t^{-2}$$

$$\frac{dx}{dt}\left(t^{-2}\right) = -2t^{-3}$$

$$\frac{dy}{dx} = \frac{-3t^{-4} + t^{-2}}{-2t^{-3}} = \frac{3t^{-4} - t^{-2}}{2t^{-3}} = \frac{3t^{-1} - t}{2} \frac{\times t}{\times t} = \frac{3 - t^2}{2^t}$$

**(b)** Find the coordinates of the point on the curve at which the tangent to the curve is parallel to the line $4y + x = 1$. [3]

$4y + x = 1$

$4y = 1 - x$

$y = \frac{-x}{4} + \frac{1}{4}$

$m = -\frac{1}{4}$

$\frac{3 - t^2}{2t} = -\frac{1}{4}$

$3 - t^2 = \frac{-2t}{4}$

$12 - 4t^2 + 2t = 0$

$t = 2$ $\quad \underbrace{t = -\frac{3}{2}}$ — t is >0 ∴ cannot be this

$\therefore t = 2$

$x = \frac{1}{2^2} = \frac{1}{4}$

$y = \frac{1}{2^3} - \frac{1}{2} = \frac{-3}{8}$

$\left(\frac{1}{4}, -\frac{3}{8}\right)$

Find the cartesian equation of the curve. Give your answer in factorised form. [3]

$x = \frac{1}{t^2} \quad y = \frac{1}{t^3} - \frac{1}{t}$

$\hookrightarrow xt^2 = 1$

$t^2 = \frac{1}{x}$

$t = \pm \frac{1}{\sqrt{x}}$

$\hookrightarrow t = +\frac{1}{\sqrt{x}}$

$(t > 0)$

$y = \frac{1}{\left(\frac{1}{\sqrt{x}}\right)^3} - \frac{1}{\left(\frac{1}{\sqrt{x}}\right)}$

$y = \frac{(\sqrt{x})^3}{1} - \sqrt{x}$

$y = x\sqrt{x} - \sqrt{x}$

$y = \sqrt{x}(x - 1)$

**8** A function is defined by $f(x) = x^3 - x$.

**(a)** By considering $\frac{f(x+h) - f(x)}{h}$, show from first principles that $f'(x) = 3x^2 - 1$. [4]

$$\frac{(x+h)^3 - (x+h) - (x^3 - x)}{h}$$

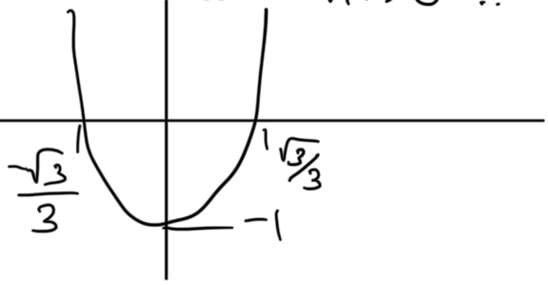
$$\hookrightarrow \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3 + x - x + h}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3 + x - (x+h)}{h} = \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$

$$= 3x^2 + 3xh + h^2 - 1$$

$h \to 0 \quad \therefore \quad 3x^2 - 1$

**(b)** Sketch the gradient function $f'(x)$. [2]



**(c)** Show that the curve $y = f(x)$ has a single point of inflection which is not a stationary point. [3]

$f''(x) = 6x$

$x = 0 \quad \therefore \quad 6x = 0$

| | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $f''(x)$ | $-6$ | $0$ | $6$ |

The second differential of the minimum value of $f'(x)$ is zero. When you take values either side of this, you get a change of sign which indicates there is a non stationary point of inflection

[5]

**9** Douglas wants to construct a model for the height of the tide in Liverpool during the day, using a cosine graph to represent the way the height changes.

He knows that the first high tide of the day measures 8.55 m and the first low tide of the day measures 1.75 m.

Douglas uses $t$ for time and $h$ for the height of the tide in metres. With his graph-drawing software set to degrees, he begins by drawing the graph of $h = 5.15 + 3.4\cos t$.

**(a)** Verify that this equation gives the correct values of $h$ for the high and low tide. [1]

$\cos t = 1$

$h = 5.15 + 3.4$

$h = 8.55$

$\cos t = -1$

$h = 5.15 - 3.4$

$h = 1.75$

Douglas also knows that the first high tide of the day occurs at 1 am and the first low tide occurs at 7.20 am. He wants $t$ to represent the time in hours after midnight, so he modifies his equation to $h = 5.15 + 3.4\cos(at + b)$.

**(b)** (i) Show that Douglas's modified equation gives the first high tide of the day occurring at the correct time if $a + b = 0$. [1]

$8.55 = 5.15 + 3\cos(at + b)$

$3.4 = 3.4\cos(at+b)$

$a = -b$

$0 = at - a$

$1 = t$

(ii) Use the time of the first low tide of the day to form a second equation relating $a$ and $b$. [1]

$1.75 = 5.15 + 3\cos(at + b)$

$-3.4 = 3.4\cos(at+b)$

$-1 = \cos(at + b)$

$180 = at + b$

$180 = \frac{22}{3}a + b$

(iii) Hence show that $a = 28.42$ correct to 2 decimal places. [2]

$a = -b$

$180 = \frac{22a}{3} - a$

$180 = a\left(\frac{22}{3} - 1\right)$

$28.42 = a$

**(c)** Douglas can only sail his boat when the height of the tide is at least 3 m.

Use the model to predict the range of times that morning when he cannot sail. [3]

$h = 5.15 + 3.4\cos(28.42t - 28.42)$

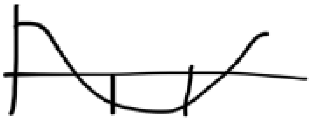
$3 \leqslant 5.15 + 3.4\cos(28.42t - 28.42)$

$5.55 \leqslant t$

$28.42t - 28.42 = 230.776, 129.2234$

$t = 9.12$

$t = 5.5$



$\therefore 5.55 \leq t \leq 9.12$ he cannot ride

So he doesn't sail between 5:33am and 9:08am

**(d)** The next high tide occurs at 12.59 pm when the height of the tide is 8.91 m.

Comment on the suitability of Douglas's model. [2]

$$h = 5.15 + 3.4\cos\left(\left(28.42 \times \frac{779}{60}\right) - 28.42\right)$$

$h = 8.36$

$\therefore$ it is not suitable as this height of 8.91 is higher than the max height predicted by model